```
--------- THE LEAN PROOF ASSISTANT ---------

-- PRESENTER --
def Name := (Option String × String)
def presenter : Name :=
  (some "Fynn", "Godau")

-- for aesthetics
def day   { n : Nat } (x : Fin (n+1)) := x - 1
def month { n : Nat } (x : Fin (n+1)) := x - 1
def year : Nat → Nat := id

-- PRESENTATION DATE --
def Date  : Type := (Fin 31 × Fin 12 × Int)
def today : Date :=
  (day     26,
   month ⟨(Nat.succ Nat.zero), by simp⟩,
   year    2026)
```

Lean is a functional programming and proof language.

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Lean's logic implements the Calculus of Inductive Construction.
What does this mean?

Existence proof

How would you prove that a function of the following type exists?

$$\alpha \times (\alpha \to \beta) \to \beta$$

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By implementing it!

$$(a, f) \mapsto f(a)$$

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This is an example of the **Curry-Howard-Isomorphism**.

Curry-Howard-Isomorphism

- utilizes similarities between the worlds of functions and proofs

data type (`Type`)	$\leftrightsquigarrow$	proposition (`Prop`)
data instance	$\leftrightsquigarrow$	proof
function (of type $\alpha \to \beta$)	$\leftrightsquigarrow$	implication (of type `Prop`$\to$`Prop`)
$\times$	$\leftrightsquigarrow$	$\wedge$

- enables program code and proof code to share a single syntax

Lean

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- Curry-Howard-Isomorphism is the idea of the **Calculus of Construction** (CoC)
- Add inductive types to get the **Calculus of Inductive Construction** (CIC)
- Lean **implements** the Calculus of Inductive Construction

Well-known examples for Inductive Types

```
inductive IntList
| nil : IntList
| cons (h : Int) (t : IntList) : IntList

inductive Nat
| zero : Nat
| succ (n : Nat) : Nat
```

Same as other functional programming languages.

Quiz

Quiz: Which of these propositions have an instance in Lean?

- `True`
- `False`
- `¬(p∧¬p)`
- `¬¬p→p`

Quiz / Answers

See `Quiz.lean`.

```
example : True := True.intro

-- type is empty; has no sound proof
example : False := sorry

example : ¬(p ∧ (¬p)) :=
  fun ⟨ a, b ⟩ => b a
```

(Note that `And` is an inductively defined type.)

Quiz / Classical Logic

See `Quiz.lean`.

```
example : ((p → False) → False) → p
  := fun f => sorry
-- we have f : (p → False) → False,
-- but need to procure an x : p
```

- intuitively, the statement holds
- no constructive proof possible

$\Rightarrow$ we need "classical reasoning", which is not available a priori

Axiom of Choice

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Using this axiom introduces classical logic into Lean.

"set α is not empty" $\Rightarrow$ "there exists an $a \in \alpha$"

$$\neg\neg p \Rightarrow p$$

Tactic mode

See `Tactics.lean`.

```
theorem linearEq :
  (a b c : Int) → (a - b < c) → (c + b > a) :=
  sorry

-- double negation and excluded middle are equivalent
theorem doubleNegationEqEm :
  (¬¬p ↔ p) ↔ (p ∨ ¬p) :=
  sorry
```

Tactic mode / Answers

```
theorem linearEq :
  (a b c : Int) → (a - b < c) → (c + b > a) :=
  by omega -- ILP solver

-- double negation and excluded middle are equivalent
theorem doubleNegationEqEm :
  (¬¬p ↔ p) ↔ (p ∨ ¬p) :=
  by grind -- find proof automatically
```

`InsertionSort` comparison

See `SortingList.lean` and `SortingArray.lean`.

List	Array
automatic termination proof	manual termination proof
—	in-place*
correctness proofs straight-forward	correctness proofs tricky

* Lean automatically optimizes the `Array.swap` operation to run in-place if possible, but the in-place property cannot be proven in the Lean language.

Comparison to Rocq

Lean and Rocq both...

- are concrete implementations of the same theory (Calculus of Inductive Construction).
- hold similar axioms.

The differences are in the "implementation details".

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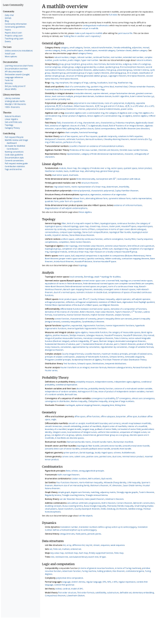
(Over which a real type theorist would fight culture wars.)

mathlib

- designed for research
- aims to be *the* coherent mathematics library for Lean
- still work in progress; some areas of math have not yet seen much progress


Lean Community

A mathlib overview

General algebra

Linear algebra

Topology

Analysis

Probability Theory

Geometry

Combinatorics

Dynamics

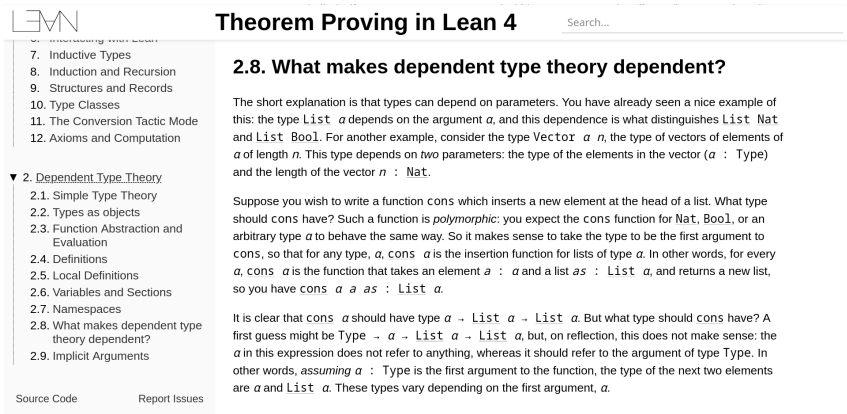
Data structures

Logic and computation

Example:
"Continuum Hypothesis is independent of ZFC" was *first* verified in Lean, using mathlib.

How to learn Lean

"Theorem Proving in Lean 4"
https://lean-lang.org/theorem_proving_in_lean4/



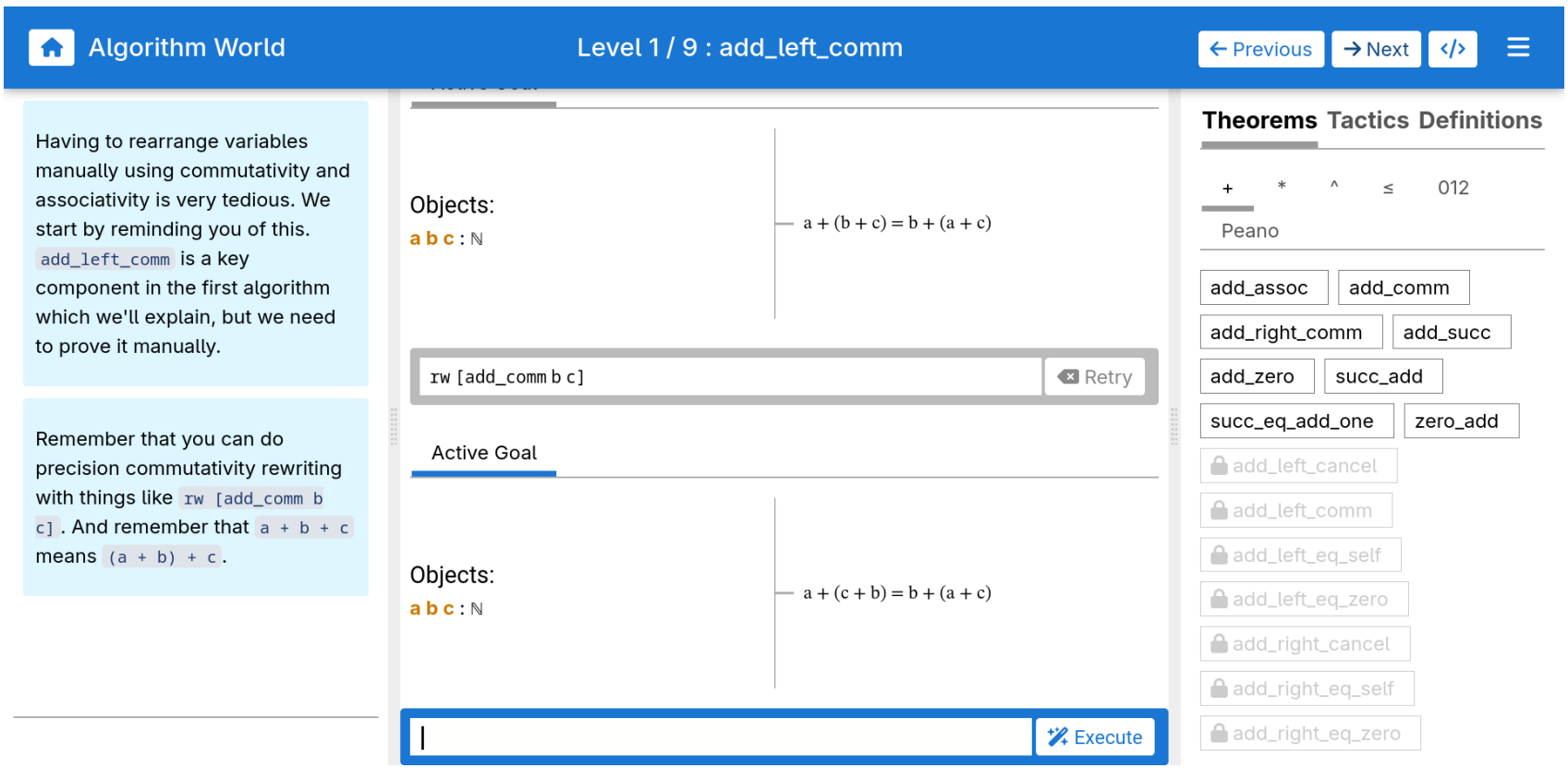
- my personal starting point, until I felt confident enough to implement and prove `insertionSort`
- to be read at a slow pace
- good iterative explanations of Lean's core concepts
- much additional type-theoretic understanding needed to *really get it*

How to learn Lean

Lean Game Server

https://adam.math.hhu.de/

- interactive tutorials teaching tactic mode
- learned about it too late, but seems like an easy way to start



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- functional programming
  $\Rightarrow$ a few hours of confusion, then the concepts become natural
- imperative programming verification
  $\Rightarrow$ pure joy not having to think about the heap
- Rocq
  $\Rightarrow$ not much will be different

~ Appendix ~

Dependent types

What is p in the last quiz example?

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example : ((p → False) → False) → p
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Its use is actually short for:

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example : (p : Prop) → ((p → False) → False) → p
  := fun p => fun f => sorry
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In the type declaration, the variable p is assigned and then used in later parts of the definition.

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In the type declaration, the variable p is assigned and then used in later parts of the definition.

This is also possible for functions on data, for polymorphic functions:

```
def insertionSort :
  (α : Type) → List α → List α := sorry
```

Axioms in Lean

- Propositional extensionality: the concept of "proof irrelevance".

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Why is this desirable?

Axioms in Lean

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```
axiom propext (a b : Prop) : (a ↔ b) → a = b
```

Consider a data type that contains a proof:

```
inductive Z₁₀
| intro (i : Nat) (proof : i < 10)

def a : Z₁₀ := Z₁₀.intro 2 (by omega)
def b : Z₁₀ := Z₁₀.intro 2 sorry

example : a = b := rfl
```

Practical motivation: proofs are erased at compile-time.

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  Using this axiom introduces classical logic into Lean.

$$\text{“set } \alpha \text{ is not empty”} \Rightarrow \text{“there exists an } a \in \alpha\text{”}$$

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Axioms in Lean and Rocq

- Lean's axioms imply Rocq's axioms
- Rocq's axioms do not imply definitional proof irrelevance

Computability of functions

`noncomputable` functions:

- for reasoning
- can use axioms
- are dropped at runtime

computable functions:

- for evaluation (executing)
- can be compiled as standalone programs

Lean tracks each function's status.

`partial` functions

- Lean requires a termination proof by default
- no termination proof necessary for `partial` or `partial_fixpoint` functions
- `partial` functions are opaque – cannot be unfolded
- `partial_fixpoint` functions are not opaque, but require recursion to be in tail position
- a proof that return type is inhabited (`Nonempty`) is always required, due to soundness